

# The ternary Goldbach problem with primes in positive density sets

Quanli Shen \*

## Abstract

Let  $\mathcal{P}$  denote the set of all primes.  $P_1, P_2, P_3$  are three subsets of  $\mathcal{P}$ . Let  $\underline{\delta}(P_i)$  ( $i = 1, 2, 3$ ) denote the lower density of  $P_i$  in  $\mathcal{P}$ , respectively. It is proved that if  $\underline{\delta}(P_1) > 5/8$ ,  $\underline{\delta}(P_2) \geq 5/8$ , and  $\underline{\delta}(P_3) \geq 5/8$ , then for every sufficiently large odd integer  $n$ , there exist  $p_i \in P_i$  such that  $n = p_1 + p_2 + p_3$ . The condition is the best possible.

**Keywords.** the ternary Goldbach problem; positive density; transference principle.

## 1 Introduction

The ternary Goldbach conjecture states that every odd positive integer greater than 5 can be written as sums of three primes. It was first proposed from an exchange of letters between Goldbach and Euler in 1742. Until 1923, Hardy and Littlewood [11] claimed it is true for sufficiently large positive odd integers, depending on the generalised Riemann hypothesis (GRH). Instead, in 1937, I. M. Vinogradov [7] showed for the first time a nontrivial estimate of exponential sums over primes, and solved this problem unconditionally. It should be noted that, recently, H. A. Helfgott [8, 9, 10] (2014) has completely proved the ternary Goldbach conjecture for every odd integer  $n$  greater than 5.

The main idea used above is circle method which is founded by Hardy and Littlewood. On the other hand, B. Green proposed the transference principle, and now it is playing an increasing important role in number theory [2, 3]. Employing this method, H. Li and H. Pan extended [4] (2010) the Vinogradov's three primes theorem to a density version. Let  $\mathcal{P}$  denote the set of all primes. For a subset  $A \subset \mathcal{P}$ , the lower density of  $A$  in  $\mathcal{P}$  is defined by

$$\underline{\delta}(A) = \liminf_{N \rightarrow \infty} \frac{|A \cap [1, N]|}{|\mathcal{P} \cap [1, N]|}.$$

They stated that if  $P_1, P_2, P_3$  are three subsets of  $\mathcal{P}$  satisfying that

$$\underline{\delta}(P_1) + \underline{\delta}(P_2) + \underline{\delta}(P_3) > 2,$$

---

\*The research was supported by 973Grant 2013CB834201.

then for every sufficiently large odd integer  $n$ , there exist  $p_i \in P_i$  ( $i = 1, 2, 3$ ) such that  $n = p_1 + p_2 + p_3$ .

Motivated by the work of Li and Pan, X. Shao proved [5] (2014) that if  $A$  is a subset of  $\mathcal{P}$  with

$$\underline{\delta}(A) > \frac{5}{8},$$

then for every sufficiently large odd integer  $n$ , there exist  $p_i \in A$  ( $i = 1, 2, 3$ ) such that  $n = p_1 + p_2 + p_3$ . It is worth mentioning that X. Shao gave [6] (2014) an l-function-free proof of Vinogradov's three primes theorem.

This paper is to revise Shao's method, and show the following result.

**Theorem 1.1** *Let  $P_1, P_2, P_3$  be three subsets of  $\mathcal{P}$ , satisfying that*

$$\underline{\delta}(P_1) > \frac{5}{8}, \underline{\delta}(P_2) \geq \frac{5}{8}, \underline{\delta}(P_3) \geq \frac{5}{8}.$$

*Then for every sufficiently large odd integer  $n$ , there exist  $p_i \in P_i$  ( $i = 1, 2, 3$ ) such that  $n = p_1 + p_2 + p_3$ .*

Note that Theorem 1.1 in [5] can be immediately obtained from the above theorem. We remark that the condition in Theorem 1.1 cannot be improved, and the counterexample can be seen in [5]. Here we provide another counterexample. Let  $P_1 = P_2 = P_3 = \{n \in P | n \equiv 1, 4, 7, 11, 13 \pmod{15}\}$ . Note that  $\underline{\delta}(P_1) = \underline{\delta}(P_2) = \underline{\delta}(P_3) = 5/8$ , but  $N \equiv 2 \pmod{15}$  cannot be written by  $p_1 + p_2 + p_3$  with  $p_i \in P_i$  ( $i = 1, 2, 3$ ).

The key to our proof is the following theorem:

**Theorem 1.2** *Let  $n \geq 6$  be an even number. Let  $\{a_i\}, \{b_i\}, \{c_i\}$  ( $0 \leq i < n$ ) are three decreasing sequences of real numbers in  $[0, 1]$ . Let  $A, B, C$  denote the averages of  $\{a_i\}, \{b_i\}, \{c_i\}$ , respectively. Suppose that for all triples  $(i, j, k)$  with  $0 \leq i, j, k < n$  and  $i + j + k \geq n$ , we have*

$$a_i b_j + b_j c_k + c_k a_i \leq \frac{5}{8}(a_i + b_j + c_k).$$

*Then we have*

$$AB + BC + CA \leq \frac{5}{8}(A + B + C).$$

It was [5, Lemma 2.2] with the condition  $n \geq 10$ , which could only deduce Theorem 1.1 with  $P_i = A$  ( $i = 1, 2, 3$ ). X. Shao remarked there exists the numerical evidence for the condition  $n \geq 6$ . In this paper, we verify its truth and apply it as the critical step which enables the argument of Shao to be valid for the general case.

Theorem 1.2 can deduce the following

**Theorem 1.3** *Let  $0 < \delta < 5/32$  and  $0 < \eta < 2\delta/5$  be parameters. Let  $m$  be a square-free positive odd integer. Let  $f_1, f_2, f_3 : \mathbb{Z}_m^* \rightarrow [0, 1]$  be functions satisfying*

$$\frac{1}{\phi(m)} \sum_{x \in \mathbb{Z}_m^*} f_1(x) > \frac{5}{8} + \delta, \frac{1}{\phi(m)} \sum_{x \in \mathbb{Z}_m^*} f_2(x) > \frac{5}{8} - \eta, \frac{1}{\phi(m)} \sum_{x \in \mathbb{Z}_m^*} f_3(x) > \frac{5}{8} - \eta,$$

where  $\phi$  is the Euler totient function. Then for any  $x \in \mathbb{Z}_m$ , there exist  $a, b, c \in \mathbb{Z}_m^*$  with  $x = a + b + c$  such that

$$f_1(a)f_2(b)f_3(c) > 0, \quad f_1(a) + f_2(b) + f_3(c) > \frac{3}{2}.$$

Theorem 1.3 is crucial for applying transference principle in section 4. It also asserts that  $A + B + C$  must cover all residue classes modulo  $m$  for any square-free odd  $m$ , provided that  $A, B, C \subset \mathbb{Z}_m^*$  with  $\delta(A) > 5/8, \delta(B) \geq 5/8, \delta(C) \geq 5/8$ , where  $\delta(A)$  denotes the density of  $A$  in  $\mathbb{Z}_m^*$ . It is the following Corollary 1.4, which extends [5, Corollary 1.5]. Note that if  $m$  is a prime, Corollary 1.4 can be immediately proved by the Cauchy-Davenport-Chowla theorem [12], which asserts that if  $A, B, C$  are subsets of  $\mathbb{Z}_p$  for prime  $p$ , then  $|A + B + C| \geq \min(|A| + |B| + |C| - 2, p)$ . However, we cannot assure whether the Cauchy-Davenport-Chowla theorem is valid for arbitrary integer  $m$ .

If  $A, B, C \subset \mathbb{Z}_m^*$  are subsets of  $\mathbb{Z}_m^*$ , denote by  $f_i(x)$  ( $x = 1, 2, 3$ ) the characteristic functions of  $A, B, C$ , respectively. Then by Theorem 1.3 we have the following

**Corollary 1.4** *Let  $m$  be a square-free positive odd integer. Let  $A_1, A_2, A_3$  be three subsets of  $\mathbb{Z}_m^*$  with  $|A_1| > \frac{5}{8}\phi(m)$ , and  $|A_i| \geq \frac{5}{8}\phi(m)$  ( $i = 2, 3$ ). Then  $A_1 + A_2 + A_3 = \mathbb{Z}_m$ .*

## 2 Proof of Theorem 1.2

We first make the change of the variables  $x_i = \frac{16}{5}a_i - 1$ ,  $y_i = \frac{16}{5}b_i - 1$ ,  $z_i = \frac{16}{5}c_i - 1$ . Note that  $\{x_i\}, \{y_i\}, \{z_i\}$  are three decreasing sequences of real numbers in  $[-1, 2.2]$ . Let  $X, Y, Z$  denote the averages of  $\{x_i\}, \{y_i\}, \{z_i\}$ , respectively.

Now our goal is to confirm that if

$$x_i y_j + y_j z_k + z_k x_i \leq 3 \tag{1}$$

for all  $0 \leq i, j, k < n$  with  $i + j + k \geq n$ , then

$$XY + YZ + ZX \leq 3.$$

Write  $n = 2m$  and

$$X_0 = \sum_{i=0}^{m-1} x_i, X_1 = \sum_{i=m}^{2m-1} x_i, Y_0 = \sum_{i=0}^{m-1} y_i, Y_1 = \sum_{i=m}^{2m-1} y_i, Z_0 = \sum_{i=0}^{m-1} z_i, Z_1 = \sum_{i=m}^{2m-1} z_i.$$

Define  $\mathcal{M} = \{(i, j, k) | 0 \leq i, j < m, m \leq k \leq n-1, i + j + k \equiv 0 \pmod{m}\}$ . Note that all of the elements in  $\mathcal{M}$  except  $(0, 0, m)$  satisfy (1), and  $\#(\mathcal{M}) = m^2$ . We have

$$\sum_{(i,j,k) \in \mathcal{M}} (x_i y_j + y_j z_k + z_k x_i) - (x_0 y_0 + y_0 z_m + z_m x_0) \leq 3(m^2 - 1).$$

Noting also that if two of the variables  $i, j, k$  are fixed, then the third is uniquely determined by the condition  $i + j + k \equiv 0 \pmod{m}$ . Thus, we have

$$\sum_{(i,j,k) \in \mathcal{M}} (x_i y_j + y_j z_k + z_k x_i) = X_0 Y_0 + Y_0 Z_1 + Z_1 X_0.$$

It follows that

$$X_0 Y_0 + Y_0 Z_1 + Z_1 X_0 \leq 3(m^2 - 1) + (x_0 y_0 + y_0 z_m + z_m x_0).$$

Similarly,

$$\begin{aligned} X_0 Y_1 + Y_1 Z_0 + Z_0 X_0 &\leq 3(m^2 - 1) + (x_0 y_m + y_m z_0 + z_0 x_0), \\ X_1 Y_0 + Y_0 Z_0 + Z_0 X_1 &\leq 3(m^2 - 1) + (x_m y_0 + y_0 z_0 + z_0 x_m). \end{aligned}$$

By the above three inequalities, we claim that

$$\begin{aligned} n^2(XY + YZ + ZX) &= (X_0 + X_1)(Y_0 + Y_1) + (Y_0 + Y_1)(Z_0 + Z_1) + (Z_0 + Z_1)(X_0 + X_1) \\ &\leq 9(m^2 - 1) + (x_0 y_0 + y_0 z_m + z_m x_0) + (x_0 y_m + y_m z_0 + z_0 x_0) + \\ &\quad + (x_m y_0 + y_0 z_0 + z_0 x_m) + X_1 Y_1 + Y_1 Z_1 + Z_1 X_1. \end{aligned} \quad (2)$$

For convenience, write

$$\begin{aligned} U &= (x_0 + x_m)(y_0 + y_m) + (y_0 + y_m)(z_0 + z_m) + (z_0 + z_m)(x_0 + x_m), \\ \Delta_0 &= x_0 y_0 + y_0 z_0 + z_0 x_0, \\ \Delta_m &= x_m y_m + y_m z_m + z_m x_m, \\ \Delta_{m,0} &= x_m z_0 + y_m z_0 + y_m x_0 + z_m x_0 + x_m y_0 + z_m y_0. \end{aligned}$$

Then (2) can be denoted by

$$n^2(XY + YZ + ZX) \leq 9(m^2 - 1) + U - \Delta_m + X_1 Y_1 + Y_1 Z_1 + Z_1 X_1. \quad (3)$$

It follows from (1) that

$$\begin{aligned} x_m y_m + y_m z_0 + z_0 x_m &\leq 3, \\ x_m y_0 + y_0 z_m + z_m x_m &\leq 3, \\ x_0 y_m + y_m z_m + z_m x_0 &\leq 3. \end{aligned}$$

Then we have

$$\Delta_{m,0} \leq 9 - \Delta_m. \quad (4)$$

Together with (2), we have

$$n^2(XY + YZ + ZX) \leq 9m^2 + \Delta_0 - \Delta_m + X_1 Y_1 + Y_1 Z_1 + Z_1 X_1. \quad (5)$$

In fact, we will apply inequalities (3) and (5) repeatedly later.

Define  $\mathcal{M}' = \{(i, j, k) : m \leq i, j, k < n, i + j + k \equiv 0 \pmod{m}\}$ . It follows from (1) that

$$\sum_{(i,j,k) \in \mathcal{M}'} (x_i y_j + y_j z_k + z_k x_i) - (x_m y_m + y_m z_m + z_m x_m) \leq 3(m^2 - 1).$$

As has been done previously, we can deduce that

$$\begin{aligned} & X_1 Y_1 + Y_1 Z_1 + Z_1 X_1 - \Delta_m \\ &= \sum_{(i,j,k) \in \mathcal{M}'} (x_i y_j + y_j z_k + z_k x_i) - (x_m y_m + y_m z_m + z_m x_m) \\ &\leq 3(m^2 - 1). \end{aligned} \tag{6}$$

Write  $r = x_0 + x_m$ ,  $s = y_0 + y_m$ ,  $t = z_0 + z_m$ . We may assume that  $r + s \geq 0$ ,  $s + t \geq 0$ ,  $t + r \geq 0$ . In fact, if at least one is negative, say  $r + s < 0$ , then

$$U = rs + st + tr \leq rs - 2(r + s) = (r - 2)(s - 2) - 4 \leq (-4) \times (-4) - 4 = 12. \tag{7}$$

Note that (3), (6), and (7) together can deduce  $XY + YZ + ZX \leq 3$ . It means the lemma has been true. Hence, we only need to consider the case  $r + s \geq 0$ ,  $s + t \geq 0$ ,  $t + r \geq 0$ . We can see that  $U$  is an increasing function with the variables  $r, s, t$ .

We next consider four cases.

**Case 1.** If  $X_1, Y_1, Z_1 < 0$ . Considering the inequality (3), we note that  $X_1 Y_1 + Y_1 Z_1 + Z_1 X_1$  is decreasing with the variables  $X_1, Y_1, Z_1$ . Then we have

$$\begin{aligned} & X_1 Y_1 + Y_1 Z_1 + Z_1 X_1 \\ &\leq [x_m - (m - 1)][y_m - (m - 1)] + [y_m - (m - 1)][z_m - (m - 1)] + \\ &\quad + [z_m - (m - 1)][x_m - (m - 1)] \\ &\leq 3(m - 1)^2 - 2(m - 1)(x_m + y_m + z_m) + \Delta_m. \end{aligned}$$

Since  $U$  is increasing, we have

$$\begin{aligned} U &\leq (2.2 + x_m)(2.2 + y_m) + (2.2 + y_m)(2.2 + z_m) + (2.2 + z_m)(2.2 + x_m) \\ &\leq 14.52 + 4.4(x_m + y_m + z_m) + \Delta_m. \end{aligned}$$

Together with  $\Delta_m \leq 3$  by (1), we have

$$U - \Delta_m + X_1 Y_1 + Y_1 Z_1 + Z_1 X_1 \leq 17.52 + 3(m - 1)^2 - (2m - 6.4)(x_m + y_m + z_m).$$

If  $m = 3$ , we bound the term  $x_m + y_m + z_m$  by  $2.2 \times 3$  trivially. Then

$$U - \Delta_m + X_1 Y_1 + Y_1 Z_1 + Z_1 X_1 \leq 3m^2 - 19m + 63 \leq 3m^2 + 6.$$

If  $m \geq 4$ , note that the term  $x_m + y_m + z_m$  is greater than  $-1 \times 3$ . Then

$$U - \Delta_m + X_1 Y_1 + Y_1 Z_1 + Z_1 X_1 \leq 3m^2 + 2.$$

Hence, it follows from (3) that  $XY + YZ + ZX \leq 3$  for all  $m \geq 3$ .

**Case 2.** If exactly two of  $X_1, Y_1, Z_1$  are negative, say  $X_1 < 0$ ,  $Y_1 < 0$ , and  $Z_1 \geq 0$ . Now we consider the inequality (5). Since  $Y_1 Z_1, Z_1 X_1$  are both nonpositive, we have

$$X_1 Y_1 + Y_1 Z_1 + Z_1 X_1 \leq X_1 Y_1.$$

Noting that  $X_1, Y_1 < 0$ , then  $X_1 Y_1$  is trivially bounded by  $[x_m - (m-1)][y_m - (m-1)]$ . Hence,

$$\begin{aligned} X_1 Y_1 + Y_1 Z_1 + Z_1 X_1 - \Delta_m &\leq -(m-1)(x_m + y_m) + (m-1)^2 - (y_m z_m + z_m x_m) \\ &\leq 2(m-1) + (m-1)^2 - z_m(y_m + x_m) \\ &\leq m^2 - 1 + 2 \times 2.2. \end{aligned}$$

The second inequality above holds since  $z_m \geq 0$  when  $Z_1 \geq 0$ . Together with  $\Delta_0 \leq 3 \times 2.2^2$  and (5), we have  $n^2(XY + YZ + ZX) \leq 10m^2 + 18 \leq 12m^2$  ( $m \geq 3$ ). Hence, we have  $XY + YZ + ZX \leq 3$ .

**Case 3.** If exactly one of  $X_1, Y_1, Z_1$  are negative, say  $X_1 < 0$ ,  $Y_1 \geq 0$ , and  $Z_1 \geq 0$ . And suppose at least one of  $X_1 + Y_1$  and  $X_1 + Z_1$  is negative. We may assume  $X_1 + Y_1 < 0$ . Since the term  $X_1 Y_1 + Y_1 Z_1 + Z_1 X_1 = (X_1 + Y_1)Z_1 + X_1 Y_1 \leq 0$ , we can ignore it in (5). Noting that at most two terms of  $\Delta_m$  are nonpositive, we have  $-\Delta_m \leq 2.2^2 \times 2$ . Together with  $\Delta_0 \leq 3 \times 2.2^2$ , it follows that  $n^2(XY + YZ + ZX) \leq 9m^2 + 5 \times 2.2^2 \leq 12m^2$  ( $m \geq 3$ ). This leads to  $XY + YZ + ZX \leq 3$ .

**Case 4.** If  $X_1 + Y_1$ ,  $Y_1 + Z_1$ , and  $Z_1 + X_1$  are all nonnegative. Therefore,  $x_m + y_m$ ,  $y_m + z_m$ , and  $z_m + x_m$  are all nonnegative. Noting that  $X_1 Y_1 + Y_1 Z_1 + Z_1 X_1$  is increasing with variables  $X_1, Y_1, Z_1$ , we have

$$X_1 Y_1 + Y_1 Z_1 + Z_1 X_1 \leq m^2 \Delta_m. \quad (8)$$

Write  $E = x_0 + y_0 - 5(x_m + y_m)$ ,  $F = y_0 + z_0 - 5(y_m + z_m)$ ,  $G = z_0 + x_0 - 5(z_m + x_m)$ . Four more cases are considered below:

(i) Suppose  $E, F, G$  are all negative. Note that

$$[x_0 + y_0 - 5(x_m + y_m)](z_0 - z_m) \leq 0.$$

Upon expanding, it follows that

$$(x_0 z_0 + y_0 z_0) + 5(x_m z_m + y_m z_m) \leq x_0 z_m + y_0 z_m + 5(x_m z_0 + y_m z_0).$$

Similarly, we have

$$\begin{aligned} (y_0 x_0 + z_0 x_0) + 5(y_m x_m + z_m x_m) &\leq y_0 x_m + z_0 x_m + 5(y_m x_0 + z_m x_0), \\ (z_0 y_0 + x_0 y_0) + 5(z_m y_m + x_m y_m) &\leq z_0 y_m + x_0 y_m + 5(z_m y_0 + x_m y_0). \end{aligned}$$

Combining the inequalities above, we have

$$\Delta_0 + 5\Delta_m \leq 3\Delta_{m,0}.$$

Together with (4), we have

$$\Delta_0 + 8\Delta_m \leq 27. \quad (9)$$

Noting that  $\Delta_m \leq 3$  by (1), (5), (8), and (9) together can deduce that

$$\begin{aligned} n^2(XY + YZ + ZX) &\leq 9m^2 + \Delta_0 + (m^2 - 1)\Delta_m \\ &\leq 9m^2 + (m^2 - 9)\Delta_m + 27 \\ &\leq 12m^2 \end{aligned}$$

for  $m \geq 3$ , which implies  $XY + YZ + ZX \leq 3$ .

(ii) If exactly two of  $E, F, G$  are negative, say  $E, F < 0$ , and  $G \geq 0$ . We can see that

$$[x_0 + y_0 - 5(x_m + y_m)][z_0 + x_0 - 5(z_m + x_m)] \leq 0.$$

Upon expanding, we have

$$\Delta_0 + 25\Delta_m + (x_0 - 5x_m)^2 \leq 5\Delta_{m,0},$$

which implies that  $\Delta_0 + 25\Delta_m \leq 5\Delta_{m,0}$ . Combining it with (4), we have

$$\Delta_0 + 30\Delta_m \leq 45.$$

Then we have

$$\Delta_0 + (m^2 - 1)\Delta_m \leq \frac{3(m^2 - 1)}{2} + \frac{31 - m^2}{30}\Delta_0.$$

For  $3 \leq m \leq 5$ , we have

$$\begin{aligned} \Delta_0 + (m^2 - 1)\Delta_m &\leq \frac{3(m^2 - 1)}{2} + \frac{31 - m^2}{30} \times 2.2^2 \times 3 \\ &\leq 1.1m^2 + 14 \leq 3m^2. \end{aligned}$$

For  $m \geq 6$ , we have

$$\begin{aligned} \Delta_0 + (m^2 - 1)\Delta_m &\leq \frac{3(m^2 - 1)}{2} - \frac{31 - m^2}{30} \times 2.2^2 \times 3 \\ &\leq 2m^2 - 16 \leq 3m^2. \end{aligned}$$

Together with (5) and (8), we have  $n^2(XY + YZ + ZX) \leq 12m^2$ , which leads to  $XY + YZ + ZX \leq 3$ .

(iii) If exactly one of  $E, F, G$  is negative, say  $E < 0$ ,  $F \geq 0$ , and  $G \geq 0$ . The proof is similar to the case (ii).

(iv) If  $E, F, G$  are all nonnegative. Note that  $x_my_m \leq (\frac{x_m+y_m}{2})^2$ ,  $x_0 + y_0 \geq 5(x_m + y_m)$  and  $x_m + y_m \geq 0$  by  $X_1 + Y_1 \geq 0$ , then we have

$$x_my_m \leq (\frac{x_0 + y_0}{10})^2 \leq 0.44^2.$$

Similarly, we have  $y_mz_m \leq 0.44^2$  and  $z_mx_m \leq 0.44^2$ . It implies that  $\Delta_m \leq 3 \times 0.44^2 \leq 1$ . We have trivially  $\Delta_0 \leq 3 \times 2.2^2$ . By (5) and (8), we have  $n^2(XY + YZ + ZX) \leq 10m^2 + 14 \leq 12m^2$  which implies  $XY + YZ + ZX \leq 3$ . This completes the proof.

Here we remark that for  $n \geq 6$ , the constant  $5/8$  can be slightly improved.

### 3 Proof of Theorem 1.3

The argument of the proof is similar to that in [5]. Using Theorem 1.2 we can show that

**Lemma 3.1** *Let  $0 < \delta < 5/32$  and  $0 < \eta < 2\delta/5$  be parameters. Let  $m$  be a square-free positive integer with  $(m, 30) = 1$ . Let  $f_1, f_2, f_3 : \mathbb{Z}_m^* \rightarrow [0, 1]$  satisfy*

$$\frac{1}{\phi(m)} \sum_{x \in \mathbb{Z}_m^*} f_1(x) > \frac{5}{8} + \delta, \quad \frac{1}{\phi(m)} \sum_{x \in \mathbb{Z}_m^*} f_2(x) > \frac{5}{8} - \eta, \quad \frac{1}{\phi(m)} \sum_{x \in \mathbb{Z}_m^*} f_3(x) > \frac{5}{8} - \eta.$$

*Then for every  $x \in \mathbb{Z}_m$ , there exist  $a, b, c \in \mathbb{Z}_m^*$  with  $x = a + b + c$ , such that*

$$f_1(a)f_2(b) + f_2(b)f_3(c) + f_3(c)f_1(a) > \frac{5}{8}(f_1(a) + f_2(b) + f_3(c)).$$

**Proof.** The proof will proceed by induction. First consider the base case when  $m = p$  is prime. It could prove the conclusion only for  $p \geq 11$  while  $f_1, f_2, f_3$  might be different [5, Proposition 3.1] and for  $p \geq 7$  with the constraint condition  $f_1 = f_2 = f_3$ . Now by Theorem 1.2, we are able to show the case that  $f_1, f_2, f_3$  need not to be the same for  $p \geq 7$ . Let  $a_0 \geq a_1 \geq \dots \geq a_{p-2}$  be  $p-1$  values of  $f_1(x)$  ( $x \in \mathbb{Z}_p^*$ ) in decreasing order. Similarly, define  $b_0 \geq b_1 \geq \dots \geq b_{p-2}$  for  $f_2(x)$  ( $x \in \mathbb{Z}_p^*$ ), and  $c_0 \geq c_1 \geq \dots \geq c_{p-2}$  for  $f_3(x)$  ( $x \in \mathbb{Z}_p^*$ ). Let  $A, B, C$  denote the averages of  $\{a_i\}, \{b_i\}, \{c_i\}$ , respectively. We can deduce that

$$AB + BC + CA > \frac{5}{8}(A + B + C).$$

To prove it, we make the change of the variables  $X = \frac{16}{5}A - 1$ ,  $Y = \frac{16}{5}B - 1$ , and  $Z = \frac{16}{5}C - 1$ . Then our aim is to prove  $XY + YZ + ZX > 3$  when

$$X > 1 + \frac{16}{5}\delta, \quad Y > 1 - \frac{16}{5}\eta, \quad Z > 1 - \frac{16}{5}\eta.$$



Note that

$$\begin{aligned}
& XY + YZ + ZX \\
& > 2(1 + \frac{16}{5}\delta)(1 - \frac{16}{5}\eta) + (1 - \frac{16}{5}\eta)^2 \\
& = 3 + \frac{32}{5}\delta + (\frac{16}{5})^2\eta^2 - \frac{2 \times 16^2}{5^2}\delta\eta - \frac{64}{5}\eta \\
& > 3 + (\frac{16}{5})^2\eta^2 - \frac{32^2}{5^3}\delta^2 + \frac{32}{5^2}\delta \\
& > 3 \quad (0 < \delta < \frac{5}{32}).
\end{aligned}$$

Then, by Theorem 1.2, there exist  $0 \leq i, j, k \leq p-1$  with  $i+j+k \geq p-1$ , such that

$$a_i b_j + b_j c_k + c_k a_i > \frac{5}{8}(a_i + b_j + c_k). \quad (10)$$

Define  $I, J, K \subset \mathbb{Z}_p^*$ ,

$$I = \{x : f_1(x) \geq a_i\}, \quad J = \{x : f_2(x) \geq b_j\}, \quad K = \{x : f_3(x) \geq c_k\}.$$

Since  $\{a_i\}, \{b_i\}, \{c_i\}$  are decreasing, we have

$$|I| + |J| + |K| \geq (i+1) + (j+1) + (k+1) \geq p+2. \quad (11)$$

By the Cauchy-Davenport-Chowla theorem, it follows from (11) that

$$I + J + K = \mathbb{Z}_p.$$

That means for any  $x \in \mathbb{Z}_p$ , there exist  $a \in I, b \in J, c \in K$  such that  $x = a+b+c$ . From the definition of  $I, J, K$ , we can see that

$$f_1(a) \geq a_i, \quad f_2(b) \geq b_j, \quad f_3(c) \geq c_k.$$

Write  $h(x, y, z) = xy + yz + zx - \frac{5}{8}(x + y + z)$ . Note that  $h(x, y, z)$  is increasing with variables  $x, y, z$  on the area

$$D = \{0 \leq x, y, z \leq 1 : x + y \geq \frac{5}{8}, y + z \geq \frac{5}{8}, z + x \geq \frac{5}{8}\}.$$

In fact, (10) implies  $a_i + b_j \geq \frac{5}{8}$ . Otherwise  $a_i c_k + b_j c_k \leq \frac{5}{8}c_k$ , then  $a_i b_j > \frac{5}{8}(a_i + b_j)$ . But it is impossible since  $0 \leq a_i, b_j \leq 1$ . Similarly, we have  $b_j + c_k \geq \frac{5}{8}$ , and  $c_k + a_i \geq \frac{5}{8}$ . Hence, we have

$$h(f_1(a), f_2(b), f_3(c)) \geq h(a_i, b_j, c_k) > 0,$$

which implies

$$f_1(a)f_2(b) + f_2(b)f_3(c) + f_3(c)f_1(a) > \frac{5}{8}(f_1(a) + f_2(b) + f_3(c)).$$

Now we consider  $m$  is composite and write  $m = m'p$  with  $p \geq 7$ . Noting that  $\mathbb{Z}_m \cong \mathbb{Z}_{m'} \times \mathbb{Z}_p$ , we define  $f'_i : \mathbb{Z}_{m'}^* \rightarrow [0, 1]$  ( $i = 1, 2, 3$ ) by

$$f'_i(x) = \frac{1}{p-1} \sum_{y \in \mathbb{Z}_p^*} f_i(x, y).$$

Then by induction hypothesis, for any  $x \in \mathbb{Z}_{m'}$ , there exists  $a, b, c \in \mathbb{Z}_{m'}^*$  with  $x = a + b + c$ , such that

$$f'_1(a)f'_2(b) + f'_2(b)f'_3(c) + f'_3(c)f'_1(a) > \frac{5}{8}(f'_1(a) + f'_2(b) + f'_3(c)).$$

Define  $a_0 \geq a_1 \geq \dots \geq a_{p-2}$  be  $p-1$  values of  $f_1(a, x)$  ( $x \in \mathbb{Z}_p^*$ ) in decreasing order, and similarly  $\{b_i\}$  for  $f_2(b, x)$  and  $\{c_i\}$  for  $f_3(c, x)$ . Noting that the averages of  $\{a_i\}, \{b_i\}, \{c_i\}$  are  $f'_1(a), f'_2(b), f'_3(c)$ , respectively. It follows from Theorem 1.2 that there exist  $0 \leq i, j, k \leq p-1$  with  $i + j + k \geq p-1$ , such that

$$a_i b_j + b_j c_k + c_k a_i > \frac{5}{8}(a_i + b_j + c_k).$$

Similarly, we can deduce that for any  $y \in \mathbb{Z}_p$ , there exist  $u, v, w \in \mathbb{Z}_p^*$  with  $y = u + v + w$ , such that

$$f_1(a, u)f_2(b, v) + f_2(b, v)f_3(c, w) + f_3(c, w)f_1(a, u) > \frac{5}{8}(f_1(a, u) + f_2(b, v) + f_3(c, w)).$$

This completes the proof.  $\blacksquare$

**Lemma 3.2** *Let  $f_1, f_2, f_3 : \mathbb{Z}_{15}^* \rightarrow [0, 1]$  be arbitrary functions satisfying*

$$F_1 F_2 + F_2 F_3 + F_3 F_1 > 5(F_1 + F_2 + F_3),$$

where  $F_i = \sum_{x \in \mathbb{Z}_{15}^*} f_i(x)$ . Then for every  $x \in \mathbb{Z}_{15}$ , there exist  $a, b, c \in \mathbb{Z}_{15}^*$  with  $x = a + b + c$ , such that

$$f_1(a)f_2(b)f_3(c) > 0, \quad f_1(a) + f_2(b) + f_3(c) > \frac{3}{2}.$$

**Proof.** See [5, Proposition 3.2].  $\blacksquare$

Now we deduce Theorem 1.3. First note that if the result is true for  $m$ , then it holds for any  $m'$  dividing  $m$ . So we suppose  $15|m$ . Write  $m = 15m'$ . Note that  $(m', 30) = 1$ . Since  $\mathbb{Z}_m \cong \mathbb{Z}_{m'} \times \mathbb{Z}_{15}$ , we can write  $(u, v)$  ( $u \in \mathbb{Z}_{m'}, v \in \mathbb{Z}_{15}$ ) as the arbitrary term in  $\mathbb{Z}_m$ . Define  $f'_i : \mathbb{Z}_{m'} \rightarrow [0, 1]$  ( $i = 1, 2, 3$ ) by

$$f'_i(x) = \frac{1}{\phi(15)} \sum_{y \in \mathbb{Z}_{15}^*} f_i(x, y).$$

Note that  $f'_i(x)$  ( $i = 1, 2, 3$ ) satisfy the condition of Lemma 3.1, and we can conclude that for every  $u \in \mathbb{Z}_{m'}$ , there exist  $a_1, a_2, a_3 \in \mathbb{Z}_{m'}^*$  with  $u = a_1 + a_2 + a_3$ , such that

$$f'_1(a_1)f'_2(a_2) + f'_2(a_2)f'_3(a_3) + f'_3(a_3)f'_1(a_1) > \frac{5}{8}(f'_1(a_1) + f'_2(a_2) + f'_3(a_3)). \quad (12)$$

Now define  $f_i^\# : \mathbb{Z}_{15}^* \rightarrow [0, 1]$  by

$$f_i^\#(y) = f_i(a_i, y).$$

With (12), we note that  $f_i^\#(y)$  satisfy the condition of Lemma 3.2. Thus, for every  $v \in \mathbb{Z}_{15}$ , there exist  $b_1, b_2, b_3 \in \mathbb{Z}_{15}^*$  with  $v = b_1 + b_2 + b_3$ , such that

$$f_i^\#(b_i) > 0, f_1^\#(b_1) + f_2^\#(b_2) + f_3^\#(b_3) > \frac{3}{2}.$$

Note that  $(u, v) = (a_1, b_1) + (a_2, b_2) + (a_3, b_3)$ . It follows that

$$f_i(a_i, b_i) > 0 \ (i = 1, 2, 3), f_1(a_1, b_1) + f_2(a_2, b_2) + f_3(a_3, b_3) > \frac{3}{2}.$$

This completes the proof.

## 4 Sketch of the proof of Theorem 1.1

The proof is almost same as in [5]. Therefore, we omit the details. Theorem 1.1 can be deduced from the following transference principle Proposition 4.1.

For  $f : \mathbb{Z}_N \rightarrow \mathbb{C}$ , we define the Fourier transform of  $f$  by

$$f(r) = \sum_{x \in \mathbb{Z}_N} f(x) e_N(rx), \ r \in \mathbb{Z}_N,$$

where  $e_N(y) = \exp(2\pi i y/N)$ .

**Proposition 4.1** *Let  $N$  be a sufficiently large prime. Suppose that  $\mu_i : \mathbb{Z}_N \rightarrow \mathbb{R}^+$  and  $a_i : \mathbb{Z}_N \rightarrow \mathbb{R}^+$  ( $i = 1, 2, 3$ ) are functions satisfying the majorization condition*

$$0 \leq a_i(n) \leq \mu_i(n),$$

*and the mean condition*

$$\min(\delta_1, \delta_2, \delta_3, \delta_1 + \delta_2 + \delta_3 - 1) \geq \delta$$

*for some  $\delta > 0$ , where  $\delta_i = \sum_{x \in \mathbb{Z}_N} a_i(x)$  ( $i = 1, 2, 3$ ). Suppose that  $\mu_i$  and  $a_i$  also satisfy the pseudorandomness conditions*

$$|\hat{\mu}_i(r) - \delta_{r,0}| \leq \eta, \ r \in \mathbb{Z}_N,$$

*where  $\delta_{r,0}$  is the Kronecker delta, and*

$$\|\hat{a}_i\|_q = \left( \sum_{r \in \mathbb{Z}_N} |\hat{a}_i(r)|^q \right)^{1/q} \leq M$$

for some  $2 < q < 3$  and  $\eta, M > 0$ . Then for any  $x \in \mathbb{Z}_N$ , we have

$$\sum_{y, z \in \mathbb{Z}_N} a_1(y) a_2(z) a_3(x - y - z) \geq \frac{c(\delta)}{N}$$

for some constant  $c(\delta) > 0$  depending only on  $\delta$ , provided that  $\eta \leq \eta(\delta, M, q)$  is small enough.

**Proof.** See [5, Proposition 4.1]. ■

Let  $n$  be a very large positive odd integer. The aim is to show there exist  $p_1 \in P_1$ ,  $p_2 \in P_2$ , and  $p_3 \in P_3$  such that  $n = p_1 + p_2 + p_3$ . In the case of Theorem 1.1, we note that there exist  $0 < \delta < 5/12$  and  $0 < \eta < \delta/50$  such that

$$\begin{aligned} |P_1 \cap [1, N]| &> \left(\frac{5}{8} + \delta\right) \frac{N}{\log N}, \\ |P_i \cap [1, N]| &> \left(\frac{5}{8} - \eta\right) \frac{N}{\log N} \quad (i = 2, 3), \end{aligned} \tag{13}$$

for any sufficiently large integer  $N > 0$ . Define  $f_i : \mathbb{Z}_W^* \rightarrow [0, 1]$  ( $i = 1, 2, 3$ ) by

$$f_i(b) = \max \left( \frac{3\phi(W)}{2n} \sum_{x \in P_i \cap (W\mathbb{Z} + b), x < \frac{2n}{3}} \log x - \frac{\delta}{8}, 0 \right).$$

Here  $W = \prod_{p \text{ prime}, p < z} p$ , where  $z = z(\delta)$  is a large parameter. It follows from (13) that

$$\begin{aligned} \sum_{b \in \mathbb{Z}_W^*} f_1(b) &> \left(\frac{5}{8} + \frac{3\delta}{8}\right) \phi(W), \\ \sum_{b \in \mathbb{Z}_W^*} f_i(b) &> \left(\frac{5}{8} - \left(\frac{5\eta}{4} + \frac{\delta}{8}\right)\right) \phi(W) \quad (i = 2, 3). \end{aligned}$$

Note that  $\frac{5\eta}{4} + \frac{\delta}{8} < \frac{2}{5} \times \frac{3\delta}{8}$  by  $0 < \eta < \delta/50$ . We can deduce from Theorem 1.3 that there exist  $b_1, b_2, b_3 \in \mathbb{Z}_W^*$  with  $b_1 + b_2 + b_3 \equiv n \pmod{W}$  such that

$$f_1(b_1) f_2(b_2) f_3(b_3) > 0, f_1(b_1) + f_2(b_2) + f_3(b_3) > \frac{3}{2}. \tag{14}$$

The rest part of the proof is just like the proof in [5]. Applying (14), one can confirm the mean condition in Proposition 4.1. The pseudorandomness conditions hold by Lemma 6.2 and Lemma 6.6 in [2]. The majorization condition is satisfied immediately from the definitions of  $a_i$  and  $\mu_i$ . Then the transference principle is applied, leading to Theorem 1.1. Here we want to refer readers to section 4 of [5] for further details.

**Acknowledgements.** The author would like to thank his advisor Professor Yonghui Wang specially for his constant guidance, and Wenying Chen for helpful discussions in seminar.

## References

- [1] H. Davenport, Multiplicative number theory, 3rd ed. Grad. Texts Math. 74. Springer-Verlag, New York 2000.
- [2] B. Green, Roth's theorem in the primes. Ann. Math. **161** (2005), 1609-1636.
- [3] B. Green, T. Tao, The primes contain arbitrarily long arithmetic progressions. Ann. Math. **167** (2008), 481-547.
- [4] H. Li, H. Pan, A density version of Vinogradov's three primes theorem. Forum Math. **22** (2010), 699-714.
- [5] X. Shao, A density version of Vinogradov's three primes theorem, Duke Math. J. **163** (2014), 489-512.
- [6] X. Shao, An l-function-free proof of Vinogradov's three primes theorem, Forum of Mathematics, Sigma. **2** (2014), e27.
- [7] I. M. Vinogradov, The representation of an odd number as a sum of three primes, Dokl. Akad. Nauk. SSSR. **16** (1937), 139-142.
- [8] H. A. Helfgott, Minor arcs for Goldbach's problem. arXiv preprint arXiv:1205.5252, 2012.
- [9] H. A. Helfgott, Major arcs for Goldbach's theorem. arXiv preprint arXiv:1305.2897, 2013.
- [10] H. A. Helfgott, The ternary Goldbach conjecture is true. arXiv preprint arXiv:1312.7748, 2014.
- [11] G. H. Hardy and J. E. Littlewood, Some problems of "partitio numerorum" III: On the expression of a number as a sum of primes, Acta. Math. **44** (1923), 1-70.
- [12] T. Tao and V. Vu, Additive combinatorics, volume 105 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2006.

### Contact information:

Quanli Shen  
Department of Math, Capital Normal University  
Xi San Huan Beilu 105, Beijing 100048, P.R. China,  
Email: qlshen@outlook.com.